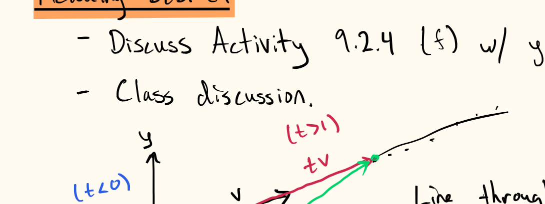
Meeting 3 | Math 22

Reading Debrief

- Discuss Activity 9.2.4 (f) w/ your group.
- Class discussion.



Line through terminal point of u in the direction v.

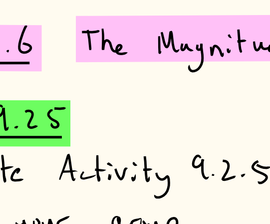
Section 9.2.6 The Magnitude of a Vector

Activity 9.2.5

- Complete Activity 9.2.5 and discuss with your group.
- Class discussion.

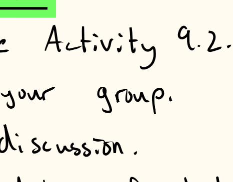
(a)–(c) Use distance formula!

(a)



$$|\overrightarrow{AB}| = \sqrt{(4-2)^2+(7-3)^2} = \sqrt{4+16} = \sqrt{20}$$

(b)



$v = \langle v_1, v_2\rangle$

$$|v| = \sqrt{v_1^2+v_2^2}$$

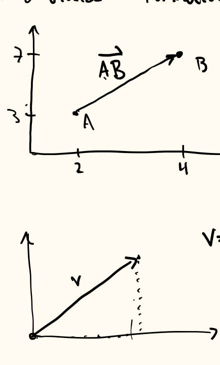
(c) $v = \langle v_1, v_2, v_3\rangle$ $\quad |v| = \sqrt{v_1^2+v_2^2+v_3^2}$

(d) $|u| = \sqrt{2^2+3^2} = \sqrt{13}$ $\quad |v| = \sqrt{(-1)^2+2^2} = \sqrt{5}$

$|u+v| = |\langle 1,5\rangle| = \sqrt{1+5^2} = \sqrt{26}$

$|u|+|v| \neq |u+v|$

(e) $|u+v| = |u|+|v|$ if and only if u and v are parallel



(f) $|2u| = \sqrt{(2\cdot 2)^2+(2\cdot 3)^2} = \sqrt{52} = 2\sqrt{13} = 2|u|$

$|3u| = \sqrt{117} = 3\sqrt{13} = 3|u|$

$|-2u| = \sqrt{52} = 2|u|$

(g) $|tu| = |t|\cdot|u|$

(h) $|i| = 1 \quad |j| = 1 \quad |i+j| = \sqrt{2}$

unit vectors.

(i) Claim: $\frac{u}{|u|}$ is a unit vector. By (g) $\left|\frac{u}{|u|}\right| \overset{(g)}{=} \left|\frac{1}{|u|}\right| |u| = 1.$

End of Section 9.2

Section 9.3 The Dot Product

Reading Debrief

- Discuss Preview Activity 9.3.1 w/ group.
- Are there any questions from Sections 9.3.1–9.3.2?

Section 9.3.1 The Dot Product

Definition 9.3.4 The dot product of $u = \langle u_1, \ldots, u_n\rangle$ and $v = \langle v_1, \ldots, v_n\rangle$ is the real number

$$u\cdot v = u_1v_1 + u_2v_2 + \cdots + u_nv_n$$

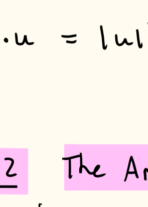
Properties of the Dot Product

For all $u, v \in \mathbb{R}^n$ and $c \in \mathbb{R}$, we have

1. $u\cdot v = v\cdot u$
2. $(u+v)\cdot w = u\cdot w + v\cdot w$
3. $(cu)\cdot v = c(u\cdot v)$
4. $u\cdot u = |u|^2$

Section 9.3.2 The Angle Between Vectors

Any two vectors u and v determine a triangle



By Law of Cosines: $|u-v|^2 = |u|^2+|v|^2-2|u||v|\cos\theta.$

OTOH: $|u-v|^2 = (u-v)\cdot(u-v) = |u|^2 - 2u\cdot v + |v|^2$

Equate and cancel: $u\cdot v = |u||v|\cos\theta$

Thus, the angle θ between u and v is

$$\theta = \cos^{-1}\left(\frac{u\cdot v}{|u||v|}\right)$$

Activity 9.3.3

- Complete Activity 9.3.3 and discuss with your group.
- Class discussion.

(a) $|u| = \sqrt{u\cdot u} = \sqrt{1\cdot 1+2\cdot 2+(-3)\cdot(-3)} = \sqrt{1+4+9} = \sqrt{14}$

(b) $\theta = \cos^{-1}\left(\frac{2}{\sqrt{5}\sqrt{17}}\right) = 77.5°$

(c) $\theta = \cos^{-1}\left(\frac{-3}{\sqrt{14}\cdot\sqrt{6}}\right) = 109.1°$

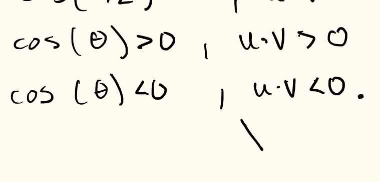
(d) Since $\cos(\pi/2) = 0$, $u\cdot v = 0$.

(e) Since $\cos(\theta) > 0$, $u\cdot v > 0$

(f) Since $\cos(\theta) < 0$, $u\cdot v < 0$.

Section 9.3.3 The Dot Product & Orthogonality

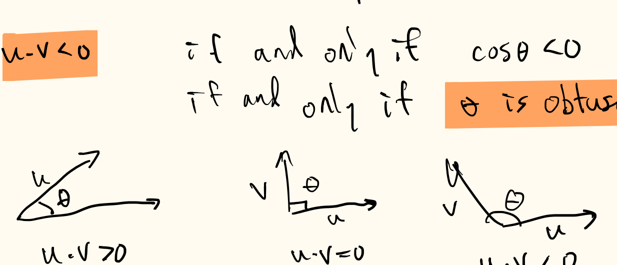
The angle between u and v $0 \le \theta \le \pi$



$u\cdot v = 0$ if and only if $\cos\theta = 0$
if and only if u, v perpendicular

$u\cdot v > 0$ if and only if $\cos\theta > 0$
if and only if θ is acute

$u\cdot v < 0$ if and only if $\cos\theta < 0$
if and only if θ is obtuse

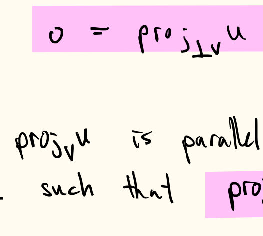


Section 9.3.5 Projections

Let $u, v \in \mathbb{R}^n$. We want to write u as a sum

$$u = \text{proj}_v u + \text{proj}_{\perp v} u$$

where $\text{proj}_v u$ is a parallel to v and $\text{proj}_{\perp v} u$ is perpendicular to v.



Since $\text{proj}_{\perp v} u$ is perp to v:

$$0 = \text{proj}_{\perp v} u \cdot v = (u - \text{proj}_v u)\cdot v = u\cdot v - \text{proj}_v u \cdot v.$$

Since $\text{proj}_v u$ is parallel to v, there exists $c \in \mathbb{R}$ such that $\text{proj}_v u = cv$. So

$$u\cdot v = \text{proj}_v u \cdot v = (cv)\cdot v = c\, v\cdot v.$$

$$\Rightarrow c = \frac{u\cdot v}{v\cdot v}$$

Thus, $\text{proj}_v u = cv = \left(\frac{u\cdot v}{v\cdot v}\right)v = \left(\frac{u\cdot v}{|v|}\right)\frac{v}{|v|}$ — The Projection of u onto v

Since $\frac{v}{|v|}$ is a unit vector, $|\text{proj}_v u| = \left|\frac{u\cdot v}{|v|}\right|$

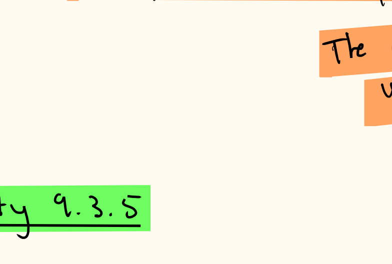
$$\text{Comp}_v u = \text{proj}_v u = \frac{u\cdot v}{|v|}$$

The Component of u along v

Activity 9.3.5

- Complete Activity 9.3.5 and discuss with your group.
- Class discussion.

(a)



$$\text{proj}_v u = \left(\frac{u\cdot v}{v\cdot v}\right)v = \frac{-40}{80}v = -\frac{1}{2}v = \boxed{\langle -2, 4\rangle}$$

$\text{comp}_v u = \frac{-40}{80}\cdot\sqrt{80} = -\sqrt{20}$

$\text{proj}_{\perp v} u = u - \text{proj}_v u = \langle 2,6\rangle - \langle -2,4\rangle = \boxed{\langle 4,2\rangle}$

(b) The projection of u onto $\text{proj}_v u$ is just $\text{proj}_v u$.

End of Section 9.3

Section 9.4.1 Computing the Cross Product

The cross product is defined on $\hat{i}, \hat{j}, \hat{k}$ using the right-hand rule

$i\times j = k$	$j\times k = i$	$k\times i = j$
$j\times i = -k$	$k\times j = -i$	$i\times k = -j$
$i\times i = 0$	$j\times j = 0$	$k\times k = 0$

It is also linear in each component

$$(au+bv)\times w = a(u\times w) + b(v\times w)$$

Let $u = \langle u_1,u_2,u_3\rangle = u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$

$v = \langle v_1,v_2,v_3\rangle = v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$

Then

$$\begin{aligned} u\times v &= (u_1\hat{i}+u_2\hat{j}+u_3\hat{k})\times(v_1\hat{i}+v_2\hat{j}+v_3\hat{k}) \\ &= u_1v_1(\underbrace{\hat{i}\times\hat{i}}_{=0}) + u_1v_2(\underbrace{\hat{i}\times\hat{j}}_{=\hat{k}}) + u_1v_3(\underbrace{\hat{i}\times\hat{k}}_{=-\hat{j}}) \\ &+ u_2v_1(\underbrace{\hat{j}\times\hat{i}}_{-\hat{k}}) + u_2v_2(\underbrace{\hat{j}\times\hat{j}}_{=0}) + u_2v_3(\underbrace{\hat{j}\times\hat{k}}_{=\hat{i}}) \\ &+ u_3v_1(\underbrace{\hat{k}\times\hat{i}}_{=\hat{j}}) + u_3v_2(\underbrace{\hat{k}\times\hat{j}}_{=-\hat{i}}) + u_3v_3(\underbrace{\hat{k}\times\hat{k}}_{=0}) \\ &= (u_2v_3-u_3v_2)\hat{i} + (u_3v_1-u_1v_3)\hat{j} + (u_1v_2-u_2v_1)\hat{k} \end{aligned}$$

Definition 9.4.4 The cross product of vectors $u = u_1\hat{i}+u_2\hat{j}+u_3\hat{k}$ and $v = v_1\hat{i}+v_2\hat{j}+v_3\hat{k}$ is the vector

$$u\times v = (u_2v_3-u_3v_2)\hat{i} - (u_1v_3-u_3v_1)\hat{j} + (u_1v_2-u_2v_1)\hat{k}.$$

Determinants The determinant of a 2x2 matrix $\begin{bmatrix} a & b \\ c & d\end{bmatrix}$ is the real number

$$\begin{vmatrix} a & b \\ c & d\end{vmatrix} = \boxed{ad} - \boxed{bc}$$

The determinant of a 3x3 matrix $\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i\end{bmatrix}$ is defined using 2x2 determinants

"Expansion about first row"

$$\begin{vmatrix} \overset{+}{a} & \overset{-}{b} & \overset{+}{c} \\ d & e & f \\ g & h & i\end{vmatrix} = a\begin{vmatrix} e & f \\ h & i\end{vmatrix} - b\begin{vmatrix} d & f \\ g & i\end{vmatrix} + c\begin{vmatrix} d & e \\ g & h\end{vmatrix}$$

- Cover up row/column containing a to get $\begin{vmatrix} e & f \\ h & i\end{vmatrix}$
- Cover up row/column containing b to get $\begin{vmatrix} d & f \\ g & i\end{vmatrix}$
- Cover up row/column containing c to get $\begin{vmatrix} d & e \\ g & h\end{vmatrix}$

You can use the same idea to expand about any row or column. Sign convention: $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & +\end{vmatrix}$

Determinants can be used to write the cross product formula in a way that's easy to remember

$$\begin{aligned} u\times v &= (u_2v_3-u_3v_2)\hat{i} - (u_1v_3-u_3v_1)\hat{j} + (u_1v_2-u_2v_1)\hat{k}. \\ &= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3\end{vmatrix}\hat{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3\end{vmatrix}\hat{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2\end{vmatrix}\hat{k} \\ &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3\end{vmatrix} \end{aligned}$$

Easy to Remember!

Activity 9.4.2

- Complete and discuss w/ your group.
- Class discussion

(a) $u\times v = \langle 0,1,3\rangle \times \langle 2,-1,0\rangle$

$$= \begin{vmatrix} \hat{i} & \hat{j} & k \\ 0 & 1 & 3 \\ 2 & -1 & 0\end{vmatrix} = \begin{vmatrix} 1 & 3 \\ -1 & 0\end{vmatrix}\hat{i} - \begin{vmatrix} 0 & 3 \\ 2 & 0\end{vmatrix}\hat{j} + \begin{vmatrix} 0 & 1 \\ 2 & -1\end{vmatrix}\hat{k}$$

$$= (0-(-3))\hat{i} - (0-6)\hat{j} + (0-2)\hat{k} = \langle 3, 6, -2\rangle$$

(b) $u\cdot(u\times v) = \langle 0,1,3\rangle\cdot\langle 3,6,-2\rangle = 0\cdot 3+1\cdot 6+3\cdot(-2) = 0$

$v\cdot(u\times v) = 0$

The cross product is perp. to both u and v.

(c) $v\times\hat{i} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 0 \\ 1 & 0 & 0\end{vmatrix}$ Expand about 3rd row

$$= 1\begin{vmatrix} \hat{j} & \hat{k} \\ -1 & 0\end{vmatrix} = \hat{j}\cdot 0 - (-\hat{k}) = \hat{k}.$$

(d) Not associative.

(e) $u\times u = 0$

Properties of the Cross Product

Let $u, v, w \in \mathbb{R}^3$ and let $c \in \mathbb{R}$. Then

1. $u\times v = -(v\times u)$
2. $(u+v)\times w = (u\times w) + (v\times w)$
3. $(cu)\times v = c(u\times v) = u\times(cv)$
4. $u\times v = 0$ if and only if u and v are parallel.
5. The cross product is not associative.